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THE DYNAMIC PROBLEM OF INTERACTION BETWEEN AN ELASTIC PUNCH AND A FLUID THROUGH A THIN COVER*

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The problem of interaction through a thin cover between an elastic body (punch) and an infinitely deep layer of perfect heavy fluid is considered. The elastic punch is pressed to the cover boundary and moves along it at constant velocity without friction. The flow of fluid is assumed stable and potential. Problems of this kind arise in investigations of processes of dynamic action of elastic bodies on the surface of an ice field. With the use of the integral Fourier transform the problem is reduced in a finite interval to an integral equation of the first kind of the convolution type with a singular kernel. The method of orthogonal polynomials /1/ is used for obtaining an approximate solution of the derived equation. The structure of solution is analyzed.

1. Statement of the problem, the integral equation. Consider a layer of perfect heavy fluid of infinite depth ($y \leqslant 0$) and density ρ, whose surface is covered by a layer of small thickness h with elastic properties G and ν. An elastic punch (G_0, ν_0), pressed with force P and moment $M = Pe$ to the boundary of this composite base, moves along it without friction at constant velocity V. It is assumed that under these conditions the cover layer does not peel off from the fluid. Let in a coordinate system attached to the punch, the punch base be defined by function $f(x')$, and the contact line by the inequality $|x'| \leqslant a$. In the approximation of the Hertz theory for the displacement derivative v_0 of points of the punch surface along the y-axis we then have the formula

$$v_0'(x', 0, t) = -\frac{1}{\pi\theta_0}\int_{-a}^{a}\frac{q^*(\xi, t)}{\xi - x'}\,d\xi \quad \left(\theta_0 = \frac{G_0}{1-\nu_0}\right) \tag{1.1}$$

where $q^*(x, t) = q(x')$ is the contact pressure that is nonzero only when $|x'| \leqslant a$, $x' = x - Vt$.

As the physical model of the cover we take that of a thin plate extended lengthwise by a continuous stress. Such layer is defined by the equation

$$h^3\beta^* v^{(4)} - h\sigma v'' = p^*(x, t) - q(x') - h\rho^* v^{\cdot\cdot}, \quad \beta^* = G[6(1-\nu)]^{-1} \tag{1.2}$$

where v is the displacement of points of the median plane of the plate along the y-axis, σ is the normal stress averaged over the thickness acting in the transverse cross section of the layer, $p^*(x, t) = p(x')$ is the reaction pressure of the fluid on the layer, and ρ^* is the density of the layer material. In what follows the prime at the moving coordinate x' will be omitted, since the analysis is carried out in the system attached to the punch.

Let us assume that the physico-mechanical properties of the fluid are defined by the linearized equations of a stabilized potential flow

$$\Delta\varphi = 0, \quad v_x = \frac{\partial\varphi}{\partial x} - V, \quad v_y = \frac{\partial\varphi}{\partial y}, \quad p = \rho V\frac{\partial\varphi}{\partial x} - \rho g y \tag{1.3}$$

where $\varphi(x, y)$ is the velocity potential, p is the fluid pressure, v_x, v_y are projections of fluid particle velocity on the axes of the moving coordinate system, g is the gravity constant, and ρ the fluid density.

The condition of the punch and cover contact for $|x| \leqslant a$ is obviously of the form

$$v + v_0 = -[\delta + \alpha^* x - f(x)] \tag{1.4}$$

where $\delta + \alpha^* x$ is the rigid displacement of the punch under the action of the applied to it force P and moment M.

Taking into account the small thickness of the plate we transfer condition (1.2) from its median plane to the fluid boundary $y = 0$. Then

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$$Dv^{(4)} - Tv'' = p(x) - q(x), \quad D = h^3\beta^*, \quad T = h(\sigma - \rho^* V^2) \tag{1.5}$$

We assume that, as in the theory of a thin wing /2/, the condition of contact between fluid and cover surface is of the form

$$\frac{\partial \varphi}{\partial y} = -V\frac{\partial v}{\partial x} \tag{1.6}$$

Then in conformity with (1.6) we can rewrite condition (1.2) in the form (when $y = 0$)

$$\frac{\partial^2}{\partial x \partial y}\left(D\frac{\partial^2 \varphi}{\partial x^2} - T\varphi\right) = V[p(x) - q(x)] \tag{1.7}$$

We assume that perturbations of the fluid induced by the punch motion disappear as $(x^2 + y^2) \to \infty$.

Let us now solve Eq. (1.3) with boundary condition (1.7) and the condition of absence of perturbations at infinity using the integral Fourier transform. We obtain the following formula for the derivative of displacement v at $y = 0$:

$$v(x, 0) = -\frac{1}{\pi D}\int_{-a}^{a} q(\xi)\, d\xi \int_0^\infty \frac{A_0 u \sin u(\xi - x)\, du}{A_0 u^4 + A_2 u^2 - A_3 u + A_4} \tag{1.8}$$

$$A_0 = D, \quad A_2 = T, \quad A_3 = \rho V^2, \quad A_4 = \rho g$$

Consider the case when the punch velocity is

$$V < V_*, \text{ where } \quad V_*^2 = \varkappa^2(\sqrt{1 + 2\varkappa^{-2}\sigma/\rho^*} - 1), \quad \varkappa^2 = 16\rho^* gh(9\rho)^{-1}.$$

The integrand of the inner integral in (1.8) has then no poles on the real semiaxis $u \geqslant 0$.

Using the condition of contact of the punch and cover we obtain an integral equation in $q(x)$ which in dimensionless variables and notation

$$\varphi(x') = q(ax')\theta_0^{-1}, \quad r(x') = f(ax')a^{-1}$$
$$x' = xa^{-1}, \quad \xi' = \xi a^{-1},$$
$$\lambda = ha^{-1}, \quad u' = uh, \quad \mu = \theta_0(\beta^*)^{-1}, \quad A_i' = A_i(A_4 h^{4-i})^{-1}$$
$$(i = 0, 2, 3)$$

is of the form

$$\int_{-1}^{1} \frac{\varphi(\xi')}{\xi' - x'}\, d\xi' + \frac{\mu}{\lambda}\int_{-1}^{1} \varphi(\xi') K\left(\frac{\xi' - x'}{\lambda}\right) d\xi' = \pi[\alpha^* - r'(x')], \quad (|x'| \leqslant 1) \tag{1.9}$$

$$K(z) = \int_0^\infty L(u')\sin u'z\, du', \quad L(u') = \frac{A_0' u'}{A_0'(u')^4 + A_2'(u')^2 - A_3' u' + 1}$$

Below, the prime at dimensionless variables is omitted.

Equation (1.9) must be supplemented by the conditions of statics

$$N_0 = P(\theta_0 a)^{-1} = \int_{-1}^{1} \varphi(\xi)\, d\xi, \quad N_1 = Pe(\theta_0 a^2)^{-1} = \int_{-1}^{1} \xi\varphi(\xi)\, d\xi \tag{1.10}$$

2. Structure of the solution of integral equation (1.9). Prior to passing to the proof of theorem on the structure of solution of the integral equation (1.9) obtained in Sect.1, let us analyze some of the properties of function $K(z)$ which will be subsequently required. Taking into account the asymptotic behavior

$$L(u) = A_0 u + O(u^2) \quad (u \to 0)$$
$$L(u) = u^{-3} + O(u^{-5}) \quad (u \to \infty) \tag{2.1}$$

we formulate the following lemma.

Lemma. Relation

$$K(z) \in B_1^1(-R, R), \quad K(z) \sim z \quad (z \to 0) \tag{2.2}$$

holds for all values of variable $z \in (-R, R)$, where R is any arbitrarily large number. Here $B_k^\alpha(-R, R)$ is the domain of functions whose k-th derivatives satisfy at $|z| \leqslant R$ Hölder's condition with exponent $0 < \alpha \leqslant 1$. When $k = \alpha = 0$, we have the domain $C(-R, R)$ of functions that are continuous on $[-R, R]$.

The lemma can be proved taking into account the following integrals /3/:

$$\int_0^\infty \frac{\sin uz}{u}\, du = \frac{\pi}{2} \operatorname{sgn} z, \quad \int_0^\infty \frac{uz - \sin uz}{u^3}\, du = \frac{\pi}{4} z^2 \operatorname{sgn} z$$

Let us recall some statements from the theory of singular integral equations with Cauchy kernel /4/ related to the following auxilliary equation:

$$\int_{-1}^{1} \frac{\varphi(\xi)}{\xi - x}\, d\xi = \pi\psi(x) \quad (|x| \leqslant 1) \tag{2.3}$$

Theorem 1. If function $\psi(x) \in B_0^\alpha(-1,1)$ and $\alpha > 0$, the solution of the integral equation (2.3) exists in the class $L_p(-1,1), 1 < p < 2$ and is of the form

$$\varphi(x) = \omega^*(x)(1 - x^2)^{-1/2} \tag{2.4}$$

$$\omega^*(x) = \frac{1}{\pi}\left[N_0 - \int_{-1}^{1} \frac{\psi(\xi)\sqrt{1-\xi^2}}{\xi - x}\, d\xi\right], \quad \omega^*(x) \in B_0^\nu(-1,1)$$

with $\nu = \alpha$ when $\alpha < 1$, and $\nu = 1-0$ when $\alpha = 1$.

If function $\psi(x) \in B_0^\alpha(-1,1)$ $(0 < \alpha \leqslant 1)$, $\psi(x) \in B_0^\beta(1-\varepsilon, 1)$ $(\varepsilon > 0,\ {}^1/_2 < \beta \leqslant 1)$ and the relation

$$N_0 + \int_{-1}^{1} \sqrt{\frac{1+\xi}{1-\xi}}\, \psi(\xi)\, d\xi = 0 \tag{2.5}$$

are satisfied, the solution of integral equation (2.3) is of the form

$$\varphi(x) = \omega^*(x)\sqrt{\frac{1-x}{1+x}}, \quad \omega^*(x) = -\frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1+\xi}{1-\xi}}\, \frac{\psi(\xi)}{\xi - x}\, d\xi \tag{2.6}$$

$$\omega^*(x) \in B_0^\nu(-1, 1), \quad \nu = \inf(\alpha, \beta - {}^1/_2)$$

If function

$\psi(x) \in B_0^\alpha(-1,1)$ $(0 < \alpha \leqslant 1)$, $\psi(x) \in B_0^\beta(1-\varepsilon, 1)$, $\psi(x) \in B_0^\beta(-1, -1+\varepsilon)$ $(\varepsilon > 0, {}^1/_2 < \beta \leqslant 1)$

and relations

$$N_0 + \int_{-1}^{1} \frac{\psi(\xi)\,\xi\, d\xi}{\sqrt{1-\xi^2}} = 0, \quad \int_{-1}^{1} \frac{\psi(\xi)\, d\xi}{\sqrt{1-\xi^2}} = 0 \tag{2.7}$$

are satisfied, the solution of integral equation (2.3) is of the form

$$\varphi(x) = \omega^*(x)\sqrt{1-x^2}, \quad \omega^*(x) = -\frac{1}{\pi}\int_{-1}^{1} \frac{\psi(\xi)\, d\xi}{\sqrt{1-\xi^2}\,(\xi - x)} \tag{2.8}$$

$$\omega^*(x) \in B_0^\nu(-1,1), \quad \nu = \inf(\alpha, \beta - {}^1/_2)$$

where $L_p(-1,1)$ $(p \geqslant 1)$ is the domain of functions that are summable with exponent p in the segment $[-1,1]$.

Formulas (2.4)—(2.8) are, thus, transforms of the integral equation (2.3) and enable us to obtain its solutions that are unbounded at both edges, bounded at one edge, or bounded at both edges.

Assuming now that $\varphi(x) \in L_p(-1,1)$ $(1 < p < 2)$ in (1.9) and using the lemma of Theorem 1 and the results of /5/, we can formulate the following theorems on the sturcture of solution of the integral equation (1.9).

Theorem 2. If $r(x) \in B_1^\alpha(-1,1)$, $0 < \alpha \leqslant 1$, then, if for given $\lambda, \mu \in (0, \infty)$ there exits a solution of Eq. (1.9) such that $\varphi(x) \in L_p(-1,1)$, $1 < p < 2$, $\varphi(x)$ is of the form

$$\varphi(x) = \omega(x)(1 - x^2)^{-1/2}, \quad \omega(x) \in B_0^\nu(-1,1) \tag{2.9}$$

and $\gamma = \alpha$ when $\alpha < 1$, and $\gamma = 1 - \varepsilon$, $\varepsilon > 0$ when $\alpha = 1$.

Theorem 3. If 1) $r(x) \in B_1^\alpha(-1,1)$, $0 < \alpha \leqslant 1$, 2) $r(x) \in B_1^\beta \cdot (1-\varepsilon, 1)$, $\varepsilon > 0$, $1/2 < \beta \leqslant 1$, then, if for given $\lambda, \mu \in (0, \infty)$ there exists a solution of Eq.(1.9) such that 1) $\varphi(x) \in L_p(-1,1)$, $1 < p < 2$, 2) $|\varphi(x)| \leqslant m$, $m > 0$ for $1 - \varepsilon \leqslant x \leqslant 1$ and condition

$$N_0 = -\pi\alpha^* + \int_{-1}^{1} \sqrt{\frac{1+\xi}{1-\xi}} [r'(\xi) + \mu\psi(\xi)]\, d\xi$$

$$\psi(\chi) = \frac{1}{\pi\lambda} \int_{-1}^{1} \varphi(\xi) K\left(\frac{\xi - x}{\lambda}\right) d\xi$$

is satisfied, $\varphi(x)$ is of the form

$$\varphi(x) = \sqrt{\frac{1-x}{1+x}}\,\omega(x), \quad \omega(x) \in B_0^\gamma(-1, 1), \quad \gamma = \inf(\alpha, \beta - 1/2) \tag{2.10}$$

Theorem 4. If

1) $r(x) \in B_1^\alpha(-1, 1)$, $0 < \alpha \leqslant 1$, 2) $r(x) \in B_1^\beta \cdot (1-\varepsilon, 1)$, $\varepsilon > 0$, $1/2 < \beta \leqslant 1$, 3) $r(x) \in B_1^\beta(-1, -1+\varepsilon)$, then, if for given $\lambda, \mu \in (0, \infty)$ there exists a solution of Eq.(1.9) such that 1) $\varphi(x) \in L_p(-1,1,)$ $1 < p < 2$, 2) $|\varphi(x)| \leqslant m$, $m > 0$ with $1 - \varepsilon \leqslant x \leqslant 1$ and $-1 \leqslant x \leqslant -1 + \varepsilon$, and the relations

$$N_0 = \int_{-1}^{1} [r'(\xi) + \mu\psi(\xi)] \frac{\xi\, d\xi}{\sqrt{1-\xi^2}}, \quad \pi\alpha^* = \int_{-1}^{1} [r'(\xi) + \mu\psi(\xi)] \frac{d\xi}{\sqrt{1-\xi^2}}$$

are satisfied, $\varphi(x)$ is of the form

$$\varphi(x) = \omega(x)\sqrt{1-x^2}, \quad \omega(x) \in B_0^\gamma(-1,1), \quad \gamma = \inf(\alpha, \beta - 1/2) \tag{2.11}$$

Proof of Theorems 2—4 is given in /5/.

3. Derivation of solution of Eq.(1.9).

For finding an approximate solution of the integral equation (1.9) we use the method of orthogonal polynomials which we base on the application of certain spectral relations for classical Chebyshev and Jacobi polynomials. We have the formulas /3/

$$\int_{-1}^{1} \frac{T_n(\xi)\, d\xi}{\sqrt{1-\xi^2}(\xi - x)} = \pi U_{n-1}(x) \qquad (n = 1, 2, \ldots) \tag{3.1}$$

$$\int_{-1}^{1} \frac{\sqrt{1-\xi^2}}{\xi - x} U_{n-1}(\xi)\, d\xi = -\pi T_n(x) \qquad (n = 1, 2, \ldots)$$

$$\int_{-1}^{1} \sqrt{\frac{1-\xi}{1+\xi}} \frac{P_n^{(1/2, -1/2)}(\xi)}{\xi - x}\, d\xi = -\pi P_n^{(-1/2, 1/2)}(x) \qquad (n = 0, 1, \ldots)$$

We pass to the analysis of the general unbounded case (2.9) of the original integral equation. We seek function $\omega(x)$ appearing in (2.9) in the form of the following series in Chebyshev's polynomial of the first kind:

$$\omega(x) = \sum_{k=0}^{\infty} a_k T_k(x), \quad a_0 = N_0\pi^{-1} \tag{3.2}$$

By virtue of the properties of $\omega(x)$ indicated in Theorem 2 and of relations

$$B_k^\nu(-1, 1) \subset L_{2,\rho}(-1, 1), \quad \|\omega\|_{L_{2,\rho}(-1,1)} = \|\omega\|_{l_2} \tag{3.3}$$

(where it is assumed that a_k are coefficients of Fourier function $\omega(x)$ in the closed in $L_{2,\rho}(-1, 1)$ orthonormal system of functions) series (3.2) converges to $\omega(x)$ in the norm of space $L_{2,\rho}(-1, 1)$, $\rho(x) = (1 - x^2)^{-1/2}$, and the respective sequence $\{a_k\} \in l_2$. The definition of domain $L_{2,\rho}(-1, 1)$, l_2 is given in /6/.

We expand function $\alpha^* - r'(x)$ and the addition to kernel $K((\xi - x)/\lambda)$, respectively, in single and double series of the form

$$\alpha^* - r'(x) = \sum_{n=1}^{\infty} b_n U_{n-1}(x) \tag{3.4}$$

$$K\left(\frac{\xi - x}{\lambda}\right) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} e_{ij}(\lambda)\, T_i(\xi)\, U_{j-1}(x)$$

Using the orthogonality conditions of Chebyshev's polynomials we obtain

$$e_{ij}(\lambda) = \frac{2\alpha_i}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} K\left(\frac{\xi - x}{\lambda}\right)\frac{\sqrt{1-x^2}}{\sqrt{1-\xi^2}}\, T_i(\xi)\, U_{j-1}(x)\, d\xi\, dx$$

$$b_i = \frac{2}{\pi}\int_{-1}^{1} [\alpha^* - r'(x)]\sqrt{1-x^2}\, U_{i-1}(x)\, dx, \quad \alpha_0 = 1, \quad \alpha_i = 2$$

By virtue of the described properties of functions $r'(x)$ and $K((\xi - x)/\lambda)$ series (3.4) uniformly converge to these functions for $|x| \leqslant 1$, $|\xi| \leqslant 1$, $\lambda \in (0, \infty)$.

Theorem 5. If function $r(x) \in B_1^{\alpha}(-1, 1)$ $(0 < \alpha \leqslant 1)$, the sequence of numbers $\{a_k\}$ of class l_2 which satisfies the infinite system of linear algebraic equations

$$a_n = b_n - \frac{\mu}{\pi\lambda} e_{0n}(\lambda) N_0 - \frac{\mu}{2\lambda}\sum_{k=1}^{\infty} e_{kn}(\lambda)\, a_k \quad (n = 1, 2, \ldots) \tag{3.5}$$

corresponds to any solution $\varphi(x)$ of class $L_p(-1, 1)$ $(1 < p < 2)$ of Eq.(1.9).

On the other hand, when function $r(x) \in B_1^{\alpha}(-1, 1)$ $(0 < \alpha \leqslant 1)$, then solution $\varphi(x) \in L_{4/3-0}(-1, 1)$ of Eq.(1.9) of form (2.9) corresponds to any solution $\{a_n\}$ of class l_2 of system (3.5).

To prove this we substitute into the integral equation (1.9), with allowance for Theorem 2 and formulas (3.3), functions $\varphi(x)$, $\alpha^* - r'(x)$, $K((\xi - x)/\lambda)$ of form (2.9) and (3.4), and calculate the integrals using the first of formulas (3.1) and the orthogonality property of Chebyshev's polynomials. As the result we have a formula whose both sides contain series in Chebyshev's polynomials of the second kind. Equating in it the coefficients in both sides at polynomials of the same number, we obtain the infinite system (3.5). Inverse transformations are readily carried out taking into account the relations /1/

$$\|\varphi(x)\|_{L_{4/3-0}(-1,1)} \leqslant m_1 \|\omega(x)\|_{L_{2,\rho}(-1,1)}, \quad \rho(x) = (1 - x^2)^{-1/2}$$

$$m_1 = \text{const}$$

Theorem 6. If function $r(x) \in B_1^{\alpha}(-1, 1)$ $(0 < \alpha \leqslant 1)$, the operator in the right-hand side of (3.5) acts in space l_2 and is there completely continuous for all values of parameters λ, $\mu \in (0, \infty)$.

Taking into account the properties of $K((\xi - x)/\lambda)$ and $r(x)$ indicated above and, also, formulas (3.3), we conclude that

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} e_{kn}^2(\lambda) < \infty, \quad \sum_{n=1}^{\infty} e_{0n}^2(\lambda) < \infty, \quad \{b_n\} \in l_2 \tag{3.6}$$

It follows from (3.6) that the operator in the right-hand side of (3.5) act in the space of sequencies l_2 and is there completely continuous for $\lambda, \mu \in (0, \infty)$. Thus the Hilbert alternative on the solvability of infinite systems /6/ is applicable to system (3.5).

Having solved the infinite system (3.5), we obtain using formulas (3.2) and (2.9) the solution of the input integral equation (1.9) for the general unbounded case.

In the case of solution of the integral equation (1.9) bounded at one edge $x = 1$ we seek function $\omega(x)$ appearing in (2.10) in the form

$$\omega(x) = \sum_{k=0}^{\infty} a_k P_k^{(1/2, -1/2)}(x), \quad a_0 = N_0\pi^{-1} \tag{3.7}$$

Note that by virtue of properties of function $\omega(x)$ (Theorem 3) and relations (3.3) series (3.7) converge to $\omega(x)$ in the norm of space $L_{2,\rho}(-1, 1)$, $\rho(x) = \sqrt{(1 - x)/(1 + x)}$, and the sequence $\{a_k\} \in l_2$.

Then expanding functions $\alpha^* - r'(x)$ and $K((\xi - x)/\lambda)$, respectively, in single and double series of the form

$$\alpha^* - r'(x) = \sum_{k=0}^{\infty} b_k P_k^{(-1/2,\,1/2)}(x) \tag{3.8}$$

$$K\left(\frac{\xi - x}{\lambda}\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e_{mn}(\lambda)\, P_m^{(1/2,\,-1/2)}(\xi)\, P_n^{(-1/2,\,1/2)}(x)$$

and using the orthogonality of Jacobi's polynomials, we obtain

$$e_{mn}(\lambda) = \frac{\alpha_m^*\alpha_n^*}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} K\left(\frac{\xi - x}{\lambda}\right) \times \sqrt{\frac{(1-\xi)(1+x)}{(1+\xi)(1-x)}}\, P_m^{(1/2,\,-1/2)}(\xi)\, P_n^{(-1/2,\,1/2)}(x)\, d\xi\, dx \tag{3.9}$$

$$b_n = \frac{\alpha_n^*}{\pi}\int_{-1}^{1} [\alpha^* - r'(x)]\sqrt{\frac{1+x}{1-x}}\, P_n^{(-1/2,\,1/2)}(x)\, dx,$$

$$\alpha_n^* = \left[\frac{(2n)!!}{(2n-1)!!}\right]^2$$

Series (3.8) uniformly converge to $\alpha^* - r'(x)$ and $K(\xi - x)/\lambda)$ for $|x| \leqslant 1$, $|\xi| \leqslant 1$, $\lambda \in (0, \infty)$.

Theorem 7. If function $r(x) \in B_1^{\alpha}(-1, 1)$ $(0 < \alpha \leqslant 1)$ and $r(x) \in B_1^{\beta}(1-\varepsilon, 1)$, $\varepsilon > 0$, $1/2 < \beta \leqslant 1$, then the sequence of numbers $\{a_k\} \in l_2$, that satisfies the infinite system of linear algebraic equations

$$a_n = \frac{\mu}{\lambda}\sum_{k=1}^{\infty} a_k(\alpha_k^*)^{-1} e_{kn}(\lambda) + \frac{\mu}{\lambda} e_{0n}(\lambda)\frac{N_0}{\pi} - b_n \quad (n = 1, 2, \ldots) \tag{3.10}$$

and the relation

$$\frac{N_0}{\pi} = \frac{\mu}{\lambda}\sum_{k=1}^{\infty} a_k(\alpha_k^*)^{-1} e_{k0}(\lambda) + \frac{\mu}{\lambda} e_{00}(\lambda) - b_0 \tag{3.11}$$

correspond to any solution $\varphi(x) \in L_p(-1, 1)$ $(1 < p < 2)$, $|\varphi(x)| \leqslant m$ $(m > 0)$ for $1 - \varepsilon \leqslant x \leqslant 1$ of Eq. (1.9).

On the other hand, if function $r(x) \in B_1^{\alpha}(-1, 1)$ $(0 < \alpha \leqslant 1)$ and $r(x) \in B_1^{\beta}(1-\varepsilon, 1)$, $\varepsilon > 0$, $1/2 < \beta \leqslant 1$ and relation (3.11) is satisfied, solution $\varphi(x) \in L_{4/3-0}(-1, 1)$, $|\varphi(x)| \leqslant m$ $(m > 0)$ of Eq. (1.9) of form (2.10) corresponds for $1 - \varepsilon \leqslant x \leqslant 1$ to any solution $\{a_n\} \in l_2$ of system (3.10).

Note that formula (3.11) is the conditions of boundedness of solution of Eq. (1.9) at the edge $x = 1$ and, after the determination of a_n $(n = 1, 2, \ldots)$ in (3.10), is used for determining the unknown half-length a of the contact region.

The proof of Theorem 7 can be carried out similarly to that of Theorem 5 with allowance for the inequality

$$\|\varphi(x)\|_{L_{4/3-0}(-1,1)} \leqslant m_2 \|\omega(x)\|_{L_{2,\rho}(-1,1)}, \quad \rho(x) = \sqrt{\frac{1-x}{1+x}}, \quad m_2 = \text{const}$$

which is established using the Hölder inequality /6/.

Note that, as in the previous case, the following theorem enables us to make conclusions regarding the solvability of a system in the domain of quadratically summable sequences for almost all values of parameters $\lambda, \mu \in (0, \infty)$.

Theorem 8. If functions $r(x) \in B_1^{\alpha}(-1, 1)$ $(0 < \alpha \leqslant 1)$ and $r(x) \in B_1^{\beta}(1-\varepsilon, 1)$, $\varepsilon > 0$, $1/2 < \beta \leqslant 1$, the operator in the right-hand side of (3.10) acts in space l_2 and is there completely continuous for all $\lambda, \mu \in (0, \infty)$. Proof of this theorem is the same as that of Theorem 6.

Having solved system (3.10), we find the solution of the input equation (1.9) bounded at the edge $x = 1$ and simultaneously determine the unknown half-length of the region of contact using formulas (3.11) and (1.10).

In conclusion we consider the case of solution of the integral equation (1.9) bounded at both edges $x = \pm 1$. We seek function $\omega(x)$ of the form

$$\omega(x) = \sum_{k=1}^{\infty} a_k U_{k-1}(x) \tag{3.12}$$

Then, using the representations

$$\alpha^* - r'(x) = \sum_{n=0}^{\infty} b_n T_n(x), \qquad K\left(\frac{\xi - x}{\lambda}\right) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} e_{mn}(\lambda)\, U_{m-1}(\xi)\, T_n(x) \tag{3.13}$$

and the condition of orthogonality of Chebyshev's polynomials, we obtain

$$e_{mn}(\lambda) = \frac{2a_n}{\pi^2} \int_{-1}^{1}\int_{-1}^{1} K\left(\frac{\xi - x}{\lambda}\right) \frac{\sqrt{1-\xi^2}}{\sqrt{1-x^2}} U_{m-1}(\xi) T_n(x)\, d\xi\, dx$$

$$b_n = \frac{a_n}{\pi} \int_{-1}^{1} [\alpha^* - r'(x)] \frac{T_n(x)}{\sqrt{1-x^2}}\, dx$$

On the basis of the properties of functions $r(x)$ and $K((\xi - x)/\lambda)$ indicated above it is possible to conclude that series (3.13) uniformly converge to them for all values of $|x| \leqslant 1$, $|\xi| \leqslant 1$, $\lambda \in (0, \infty)$.

Theorem 9. If

1) $r(x) \in B_1^\alpha(-1, 1)$ $(0 < \alpha \leqslant 1)$, 2) $r(x) \in B_1^\beta(1 - \varepsilon, 1)$, $\varepsilon > 0$, $1/2 < \beta \leqslant 1$, 3) $r(x) \in B_1^\beta(-1, -1 + \varepsilon)$,

the sequence of numbers $\{a_k\} \in l_2$ that satisfies the infinite system of linear algebriac equations

$$a_n = \frac{\mu}{2\lambda} \sum_{k=1}^{\infty} a_k e_{kn}(\lambda) + \frac{\mu}{\pi\lambda} e_{1n}(\lambda) N_0 - b_n \quad (n = 2, 3, \ldots) \tag{3.14}$$

and the relations

$$0 = \frac{\mu}{2\lambda} \sum_{k=2}^{\infty} a_k e_{k0}(\lambda) + \frac{\mu}{\pi\lambda} e_{10}(\lambda) N_0 - b_0 \tag{3.15}$$

$$\frac{2N_0}{\pi} = \frac{\mu}{2\lambda} \sum_{k=2}^{\infty} a_k e_{k1}(\lambda) + \frac{\mu}{\pi\lambda} e_{11}(\lambda) N_0 - b_1$$

corresponds to any solution $\varphi(x) \in L_p(-1, 1)$ $(1 < p < 2)$ $|\varphi(x)| \leqslant m$ $(m > 0)$ of Eq. (1.9) when $1 - \varepsilon \leqslant x < 1$ and $-1 \leqslant x \leqslant -1 + \varepsilon$.

Conversely, if

1) $r(x) \in B_1^\alpha(-1, 1)$ $(0 < \alpha \leqslant 1)$, 2) $r(x) \in B_1^\beta(1 - \varepsilon, 1)$, $\varepsilon > 0$, $1/2 < \beta \leqslant 1$, 3) $r(x) \in B_1^\beta(-1, -1 + \varepsilon)$

and relations (3.15) are satisfied, solution $\varphi(x) \in L_{4-0}(-1, 1)$, $|\varphi(x)| \leqslant m$ $(m > 0)$ of Eq. (1.9) of form (2.11), when $1 - \varepsilon \leqslant x \leqslant 1$ and $-1 \leqslant x \leqslant -1 + \varepsilon$, correspond to any solution $\{a_n\} \leqslant l_2$ of system (3.14).

Formulas (3.15) represent conditions of boundedness and, after the determination of a_n from (3.14), are used for finding the unknown half-length of the contact region a and of the quantity e.

Proof of Theorem 9 is the same as that of Theorem 5 if the inequality

$$\|\varphi(x)\|_{L_{4-0}(-1,1)} \leqslant m_3 \|\omega(x)\|_{L_{2,\rho}(-1,1)}, \quad \rho(x) = \sqrt{1 - x^2}, \quad m_3 = \text{const}$$

which can be checked using Hölder's inequalities /6/, is taken into account.

Theorem 10. If

1) $r(x) \in B_1^\alpha(-1, 1)$ $(0 < \alpha \leqslant 1)$, 2) $r(x) \in B_1^\beta(1 - \varepsilon, 1)$, $\varepsilon > 0$, $1/2 < \beta \leqslant 1$, 3) $r(x) \in B_1^\beta(-1, -1 + \varepsilon)$,

the operator in the right-hand side of (3.14) acts in l_2 completely continuously for all values of parameters $\lambda, \mu \in (0, \infty)$.

Proof of Theorems 6 and 10 is similar.

Theorem 10 implies that the infinite system (3.14) is solvable in l_2 for almost all values of parameters $\lambda, \mu \in (0, \infty)$.

Having solved that system, we obtain the solution of Eq. (1.9) using formulas (3.12) and (2.11) and simultaneously determine the quantities a and e in conformity with (3.15) and (1.10).

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